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On surfaces whose twistor lifts are harmonic sections

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Abstract

We study surfaces whose twistor lifts are harmonic sections, and characterize these surfaces in terms of their second fundamental forms. As a corollary, under certain assumptions for the curvature tensor, we prove that the twistor lift is a harmonic section if and only if the mean curvature vector field is a holomorphic section of the normal bundle. For surfaces in four-dimensional Euclidean space, a lower bound for the vertical energy of the twistor lifts is given. Moreover, under a certain assumption involving the mean curvature vector field, we characterize a surface in four-dimensional Euclidean space in such a way that the twistor lift is a harmonic section, and its vertical energy density is constant.

MSC: 53C42; 58E20

Keywords: Twistor space; Twistor lift; Twistor holomorphic surface; Vertical energy; Harmonic section

1. Introduction

For an oriented surface in an oriented four-dimensional Riemannian manifold, the *twistor lift*, which is a smooth map from the surface to the twistor space, is defined [12]. It is important to study these surfaces by their twistor lifts. In particular, a surface is said to be superminimal if its twistor lift is horizontal. For superminimal surfaces, see [5,12,13,18], for example. On the twistor space, an almost complex structure can be defined [1,12]. Then, surfaces with holomorphic twistor lifts relative to the almost complex structure are also considered (see [6,12], for example). Considering the canonical metrics of twistor spaces, surfaces whose twistor lifts are harmonic maps are studied (see [7, 11], for example). Harmonic maps are stationary points of the energy functionals between Riemannian manifolds. For sections with unit lengths, we consider the energy functional restricted to the space of all such sections and its stationary points, which are called *harmonic sections* (see Section 2). If a smooth section is a harmonic map in the usual sense, then it is a harmonic section. In this paper, we study surfaces whose twistor lifts are harmonic sections.

We characterize surfaces whose twistor lifts are harmonic sections in terms of their second fundamental forms (Theorem 5.2). The twistor lifts of the superminimal surfaces are harmonic sections. In addition, if a superminimal surface is compact, its twistor lift attains the minimum value of a restricted energy functional. Twistor lifts of surfaces with parallel second fundamental forms are also harmonic sections. Under certain assumptions for the curvature tensor, we prove that the twistor lift is a harmonic section if and only if the mean curvature vector field is a holomorphic

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section of the normal bundle (Corollary 5.4). The usual harmonicity of twistor lifts for surfaces in four-dimensional unit spheres are studied in [11]. In [7], Lagrangian surfaces in complex projective planes with harmonic twistor lifts are studied.

The vertical component of the energy of a section is called the vertical energy. By the definition of superminimality, a surface is superminimal if and only if the vertical energy of the twistor lift vanishes. Hence, it is natural to consider the following problem : *Assume that the ambient space does not admit any compact superminimal surface. Find a geometric constant $C > 0$ such that*

$$\text{(the vertical energy of the twistor lift)} \geq C,$$

and characterize the equality case. As a solution to this problem, we provide an answer for the case where the ambient space is four-dimensional Euclidean space (Theorem 6.1). Moreover, under an assumption involving the mean curvature vector field, we characterize a surface in four-dimensional Euclidean space such that the twistor lift is a harmonic section, and its vertical energy density is constant (Theorem 6.4).

In Section 2, we study sections of the sphere bundles and harmonic sections for Riemannian vector bundles. We recall the definition of twistor spaces and twistor lifts of surfaces in Section 3. In Section 4, we obtain the fundamental formulae related to twistor lifts. Surfaces whose twistor lifts are harmonic sections are considered in Section 5. In the last section, we study the energy density of twistor lifts for surfaces in Euclidean space.

2. Sections of the sphere bundles and harmonic sections

Throughout this paper, all manifolds and maps are assumed to be smooth. Let E be a Riemannian vector bundle with a fiber metric g^E and a metric connection ∇^E over an n-dimensional Riemannian manifold (M, g). We denote the tangent bundle of a manifold P by TP. Let $K^E : TE \to E$ be the connection map with respect to ∇^E. The space of all sections of E is denoted by $\Gamma(E)$. The canonical metric G on E is defined by

$$G(\zeta, \zeta) = g(p_*(\zeta), p_*(\zeta)) + g^E(K^E(\zeta), K^E(\zeta))$$

for $\zeta \in TE$, where $p : E \to M$ is the bundle projection. Note that $p : (E, G) \to (M, g)$ is a Riemannian submersion with totally geodesic fibers (see [1,20]). We call $\ker p_*$ (resp. $\ker K^E$) the vertical (resp. horizontal) subbundle of TE. For $\xi \in \Gamma(E)$, its vertical lift is denoted by ξ^v. For a vector field X on M, X^h stands for the horizontal lift of X. We note that $K^E(\xi^v) = \xi$ and $K^E(\xi_*(X)) = \nabla^E_X \xi$ for $\xi \in \Gamma(E)$ and $X \in TM$. Let ∇^G (resp. ∇) be the Levi-Civita connection of G (resp. g) on E (resp. M). The curvature form of ∇^E is denoted by R^E. We define $\hat{R}^E_{\xi,\eta}$ for $\xi, \eta \in \Gamma(E)$ by

$$g(\hat{R}^E_{\xi,\eta} X, Y) = g^E(R^E(X, Y)\xi, \eta),$$

where $X, Y \in TM$. The following equations hold at $u \in E$ (see [3]):

$$\nabla^G_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}(R^E(X, Y)u)^v,$$
$$\nabla^G_{X^h} \xi^v = \frac{1}{2}(\hat{R}^E_{u,\xi} X)^h + (\nabla^E_X \xi)^v,$$
$$\nabla^G_{\xi^v} Y^h = \frac{1}{2}(\hat{R}^E_{u,\xi} Y)^h,$$
$$\nabla^G_{\xi^v} \zeta^v = 0$$

for all $\xi, \zeta \in \Gamma(E)$ and $X, Y \in \Gamma(TM)$. Set

$$UE\ (=U(E)) := \{u \in E \mid g^E(u, u) = 1\}.$$

The set of all sections $\xi \in \Gamma(E)$ such that $\xi(M) \subset UE$ is denoted by $\Gamma(UE)$. A unit normal vector field η on UE in E is the vertical vector such that $\eta_u = u^v$ for $u \in UE$. For $\xi \in E$, we define the tangential lift ξ^t of ξ at $u \in UE$ by $\xi^t = \xi^v - g^E(\xi, u)\eta_u$. The tangential lift of a section $\xi \in \Gamma(E)$ is the vertical vector field ξ^t on UE whose value at $u \in UE$ is the tangential lift of $\xi(p(u))$. Let $\mathcal{A}$ be the shape operator of UE in E with respect to η.

Lemma 2.1. *For any vertical vector field U and any horizontal vector field X which are tangent to UE, we have $\mathcal{A}(U) = -U$ and $\mathcal{A}(X) = 0$.*

Proof. We may assume that U is the tangential lift ξ^t of $\xi \in \Gamma(E)$ and X is the horizontal lift Y^h of $Y \in \Gamma(TM)$. Fix $u \in UE$ and take a vertical curve $\bar{u} : I \to UE$ defined on an open interval I containing 0 such that $u = \bar{u}(0)$ and $\bar{u}'(0) = (\xi^t)_u$. We have

$$\mathcal{A}(\xi^t)_u = -(\nabla^G_{\bar{u}'(0)}\eta)_u = -\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)^v\bigg|_{t=0} = -(\xi^t)_u.$$

Similarly, take a horizontal curve $\bar{u}$ such that $u = \bar{u}(0)$ and $\bar{u}'(0) = (Y^h)_u$. We obtain

$$\mathcal{A}(Y^h)_u = -(\nabla^G_{\bar{u}'(0)}\eta)_u = -(\nabla^E_{(p\circ\bar{u})'(0)}\bar{u}(t))^v_u = 0. \quad \square$$

Let $\bar{\nabla}^G$ be the Levi-Civita connection of UE relative to the induced metric of (E, G).

Lemma 2.2. *For $\xi, \zeta \in \Gamma(E)$ and $X, Y \in \Gamma(TM)$, at $u \in UE$, we have*

$$\bar{\nabla}^G_{X^h}Y^h = (\nabla_X Y)^h - \frac{1}{2}(R^E(X, Y)u)^t,$$
$$\bar{\nabla}^G_{X^h}\xi^t = (\nabla^E_X\xi)^t + \frac{1}{2}(\hat{R}^E_{u,\xi}X)^h,$$
$$\bar{\nabla}^G_{\xi^t}Y^h = \frac{1}{2}(\hat{R}^E_{u,\xi}Y)^h,$$
$$\bar{\nabla}^G_{\xi^t}\zeta^t = -g^E(\zeta, u)\xi^t.$$

Proof. We prove only the last equation, since $\mathcal{A}(Z) = 0$ for any horizontal vector Z. Take a vertical curve $\bar{u} : I \to UE$, defined on an open interval I containing 0 such that $u = \bar{u}(0)$ and $\bar{u}'(0) = (\xi^t)_u$. Since $\zeta^t_u = \zeta^v_u - g^E(\zeta(p(u)), u)u^v$, we have

$$\begin{aligned}(\bar{\nabla}^G_{\xi^t}\zeta^t)_u &= \nabla^G_{\bar{u}'(0)}\zeta^t + G_u(\xi^t, \zeta^t)\eta_u \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\zeta^v_{\bar{u}(t)} - g^E(\zeta(p(\bar{u}(t))), \bar{u}(t))\bar{u}(t)^v)\bigg|_{t=0} + G_u(\xi^t, \zeta^t)\eta_u \\ &= -g^E(\zeta(p(u)), u)\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)^v\bigg|_{t=0} = -g^E(\zeta(p(u)), u)\xi^t_u,\end{aligned}$$

where we use Lemma 2.1. $\square$

We define H^{∇^E} by

$$H^{\nabla^E}(X, Y)\xi := -\nabla^E_X\nabla^E_Y\xi + \nabla^E_{\nabla_X Y}\xi$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(E)$. With respect to the Levi-Civita connections ∇ and $\bar{\nabla}^G$, from Lemma 2.2, we obtain

Lemma 2.3. *For $\xi \in \Gamma(UE)$, we have*

$$\bar{\nabla}^G_X\xi_*(Y) - \xi_*(\nabla_X Y) = \frac{1}{2}(\hat{R}^E_{u,\nabla^E_Y\xi}X)^h + \frac{1}{2}(\hat{R}^E_{u,\nabla^E_X\xi}Y)^h - (H^{\nabla^E}(X, Y)\xi)^t - \frac{1}{2}(R^E(X, Y)u)^t$$

for all $X, Y \in \Gamma(TM)$ at $x \in M$, where $u = \xi(x)$.

The rough Laplacian $\bar{\Delta}^{\nabla^E}$ of ∇^E is defined by

$$\bar{\Delta}^{\nabla^E}(\xi) = \sum_{i=1}^n H^{\nabla^E}(e_i, e_i)(\xi) = -\sum_{i=1}^n(\nabla^E_{e_i}\nabla^E_{e_i}\xi - \nabla^E_{\nabla_{e_i}e_i}\xi)$$

for $\xi \in \Gamma(E)$, where $e_1, \ldots, e_n$ is an orthonormal frame of (M, g). The torsion $\tau(\varphi)$ of a smooth map $\varphi : M_1 \to M_2$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is defined by

$$\tau(\varphi) = \sum_{i=1}^{l} (\nabla^2_{e_i} \varphi_*(e_i) - \varphi_*(\nabla^1_{e_i} e_i)),$$

where $e_1, \ldots, e_l$ is an orthonormal frame of (M_1, g_1) and ∇^i is the Levi-Civita connection of (M_i, g_i) $(i = 1, 2)$. By Lemma 2.3, we have

Lemma 2.4. *The torsion $\tau(\xi)$ of $\xi \in \Gamma(UE)$ is given by*

$$\tau(\xi) = \sum_{i=1}^{n} (\hat{R}^E_{u, \nabla^E_{e_i}\xi} e_i)^h - (\bar{\Delta}^{\nabla^E}(\xi))^t$$

at $x \in M$, where $u = \xi(x)$ and $e_1, \ldots, e_n$ is an orthonormal frame of (M, g).

We assume that M is compact. Let $\mathcal{E}$ be the energy functional defined on the space of all smooth maps from M to UE. For a smooth section $\xi \in \Gamma(UE)$, the energy $\mathcal{E}(\xi)$ is given by

$$\mathcal{E}(\xi) = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\nabla^E \xi\|^2 \mathrm{d}v_g,$$

where $\mathrm{d}v_g$ denotes the volume element of (M, g), and $\mathrm{Vol}(M)$ is the volume of (M, g). We say that $\xi \in \Gamma(UE)$ is a *harmonic section* if ξ is a stationary point of $\mathcal{E}|_{\Gamma(UE)}$. Obviously, if a smooth section is a harmonic map in the usual sense, then it is a harmonic section. The following fact is proved in [23].

Lemma 2.5. *A section $\xi \in \Gamma(UE)$ is a harmonic section if and only if the equation*

$$\bar{\Delta}^{\nabla^E}(\xi) = \|\nabla^E \xi\|^2 \xi \tag{2.1}$$

holds.

The Eq. (2.1) makes sense for noncompact manifolds. Therefore, we also say that $\xi \in \Gamma(UE)$ is a harmonic section if ξ satisfies (2.1) for noncompact cases. We see that $\xi \in \Gamma(UE)$ is a harmonic section if and only if

$$(\bar{\Delta}^{\nabla^E}(\xi))^t = 0 \tag{2.2}$$

on $\xi(M)$. In the case where $E = TM$, the harmonic sections are called harmonic vector fields. For harmonic vector fields, we refer to [14,22,23]. Note that submanifolds with harmonic sections are studied in [15,16].

3. Twistor spaces over four-dimensional Riemannian manifolds and twistor lifts of surfaces

In this section, we recall the twistor space over an oriented four-dimensional Riemannian manifold, and the twistor lift of a surface (see [1,9,12], for example). Note that the hyperbolic twistor spaces over pseudo-Riemannian manifolds with neutral metrics are also studied (see [2,4], for example). Let $(\tilde{M}, \tilde{g})$ be an oriented four-dimensional Riemannian manifold. The Hodge star operator is denoted by $*$. Since $*^2 = \mathrm{id}$ on the space of 2-forms $\Lambda^2(\tilde{M})$, we have

$$\Lambda^2(\tilde{M}) = \Lambda^2_+(\tilde{M}) \oplus \Lambda^2_-(\tilde{M}),$$

where $\Lambda^2_\pm(\tilde{M}) = \{\omega \in \Lambda^2(\tilde{M}) \mid *\omega = \pm\omega\}$. We choose an orthonormal frame $e_1, \ldots, e_4$ of $\tilde{M}$ defining the orientation of $\tilde{M}$. Let $\omega^1, \ldots, \omega^4$ be the dual frame of $e_1, \ldots, e_4$. We define the fiber metric $\hat{g}$ of $\Lambda^2(\tilde{M})$ by

$$\hat{g}(\omega^i \wedge \omega^j, \omega^k \wedge \omega^l) = \frac{1}{2}\begin{vmatrix} \tilde{g}(\omega^i, \omega^k) & \tilde{g}(\omega^i, \omega^l) \\ \tilde{g}(\omega^j, \omega^k) & \tilde{g}(\omega^j, \omega^l) \end{vmatrix}.$$

Set

$$s_1 := \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4,$$
$$s_2 := \omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^2,$$
$$s_3 := \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3.$$

Then s_1, s_2, s_3 is an orthonormal frame of $\Lambda^2_-(\tilde{M})$. We define $K_i : T\tilde{M} \to T\tilde{M}$ by $\tilde{g}(K_i(X), Y) = 2\hat{g}(s_i, X^\# \wedge Y^\#)$ and set $\Omega_i(X, Y) = g(K_i(X), Y)$ for $X, Y \in T\tilde{M}$, where $X^\#$ stands for the metric dual 1-form of $X \in \Gamma(T\tilde{M})$. Then we have $K_1(e_1) = e_2$, $K_1(e_3) = -e_4$, and so on. For local calculations, we need the following table:

	e_1	e_2	e_3	e_4
K_1	e_2	$-e_1$	$-e_4$	e_3
K_2	e_3	e_4	$-e_1$	$-e_2$
K_3	e_4	$-e_3$	e_2	$-e_1$

Moreover, we see that $-\Omega_i \wedge \Omega_i = \omega^1 \wedge \cdots \wedge \omega^4$ for $i = 1, 2, 3$. The endomorphism bundle of the tangent bundle $T\tilde{M}$ is denoted by $\mathrm{End}(T\tilde{M})$. Let Q be the vector subbundle of $\mathrm{End}(T\tilde{M})$, whose local triviality is given by K_1, K_2, K_3. Note that $K_2K_1 = K_3$, and Q is a parallel subbundle in $\mathrm{End}(T\tilde{M})$ with respect to the connection induced by the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}$. We use the same letter $\tilde{\nabla}$ for the connection of $\mathrm{End}(T\tilde{M})$ induced by $\tilde{\nabla}$. The twistor space $\mathcal{Z}$ over $\tilde{M}$ can be defined as the unit sphere bundle UQ of Q, where the fiber metric of Q is normalized such that $\|K_1\|^2 = \|K_2\|^2 = \|K_3\|^2 = 1$. The bundle projection $p : \mathcal{Z} \to \tilde{M}$ and the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$ induce the decomposition

$$T\mathcal{Z} = T^h\mathcal{Z} \oplus T^v\mathcal{Z}$$

into the horizontal subbundle $T^h\mathcal{Z}$ and the vertical subbundle $T^v\mathcal{Z}$ (see Section 2). On the twistor space $\mathcal{Z}$, an almost complex structure $J^{\mathcal{Z}}$ is defined by $J^{\mathcal{Z}}(X) = (J(p_*(X)))^h_J$ for all horizontal vectors X at $J \in \mathcal{Z}$ and $J^{\mathcal{Z}}(V) = J^v(V)$ for all vertical vectors V, where J^v is the canonical complex structure on each fiber $\simeq S^2(1)$ (=the two-dimensional unit sphere). We consider the canonical metric on $\mathcal{Z}$.

Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion from an oriented two-dimensional Riemannian manifold (M, g) into an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. The Levi-Civita connections of g and $\tilde{g}$ are denoted by ∇ and $\tilde{\nabla}$. Let $T^\perp M$ be the normal bundle of f and $\nabla^\perp$ the normal connection of $T^\perp M$. Using an orthonormal frame e_1, e_2, e_3, e_4 adapted to the orientation of $\tilde{M}$, such that e_1, e_2 defines the orientation of M and e_3, e_4 are normal to M, we define $J : TM \to TM$ by $J(e_1) = e_2$ and $J(e_2) = -e_1$, and $J^\perp : T^\perp M \to T^\perp M$ by $J^\perp(e_3) = -e_4$, and $J^\perp(e_4) = e_3$. It is easy to see that $\nabla J = 0$ and $\nabla^\perp J^\perp = 0$. We set

$$\tilde{J}(X) := J(X) \quad \text{and} \quad \tilde{J}(\zeta) := J^\perp(\zeta)$$

for $X \in TM$ and $\zeta \in T^\perp M$. Then $\tilde{J}$ is a section of $U(f^\#Q)$ $(=f^\#(\mathcal{Z}))$ and $\tilde{J}$ is called the *twistor lift* of M. Hereafter, for simplicity, we often omit the symbol "f" for induced objects of the immersion f, if there is no confusion.

4. Fundamental formulae for surfaces in four-dimensional manifolds related to twistor lifts

In this section, we prepare several fundamental formulae for surfaces in four-dimensional manifolds. Let (M, g) be an oriented surface in an oriented, four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Let α and A be the second fundamental form and the shape operator of M respectively. The mean curvature vector of M is denoted by H. We define $\nabla'\alpha$ by

$$(\nabla'_X\alpha)(Y, Z) = \nabla^\perp_X\alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. Let $\tilde{R}$, R and $R^\perp$ be the curvature forms of $\tilde{\nabla}$, ∇ and $\nabla^\perp$, respectively. Then the following equations hold

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \tilde{g}(\alpha(X, Z), \alpha(Y, W)) - \tilde{g}(\alpha(X, W), \alpha(Y, Z)), \tag{4.1}$$

$$\tilde{g}(\tilde{R}(X, Y)Z, \xi) = \tilde{g}((\nabla'_X\alpha)(Y, Z), \xi) - \tilde{g}((\nabla'_Y\alpha)(X, Z), \xi), \tag{4.2}$$

$$\tilde{g}(\tilde{R}(X, Y)\xi, \zeta) = \tilde{g}(R^\perp(X, Y)\xi, \zeta) + g(A_\xi X, A_\zeta Y) - g(A_\xi Y, A_\zeta X) \tag{4.3}$$

for all $X, Y \in TM$ and $\xi, \zeta \in T^\perp M$. If e_1, e_2, e_3, e_4 is an orthonormal frame adapted to the orientation of $\tilde{M}$ such that e_1, e_2 defines the orientation of M and e_3, e_4 are normal to M, then we have $\tilde{J} = f^\#(K_1)$. Set $I = f^\#(K_2)$ and $K = f^\#(K_3) \in \Gamma(U(f^\#Q))$.

Lemma 4.1. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then, we have*

$$\tilde{\nabla}_X \tilde{J} = \tilde{g}(\alpha(X, Je_1) - J^{\perp}\alpha(X, e_1), e_3)I + \tilde{g}(\alpha(X, Je_1) - J^{\perp}\alpha(X, e_1), e_4)K$$

for all $X \in TM$.

Proof. We have

$$\begin{aligned}\tilde{\nabla}_X \tilde{J} &= \frac{1}{4}\tilde{g}(\tilde{\nabla}_X \tilde{J}, I)I + \frac{1}{4}\tilde{g}(\tilde{\nabla}_X \tilde{J}, K)K \\ &= \frac{1}{4}\left\{\sum_{i=1}^{2}\tilde{g}((\tilde{\nabla}_X \tilde{J})(e_i), Ie_i)I + \sum_{j=3}^{4}\tilde{g}((\tilde{\nabla}_X \tilde{J})(e_j), Ie_j)I\right. \\ &\quad \left. + \sum_{i=1}^{2}\tilde{g}((\tilde{\nabla}_X \tilde{J})(e_i), Ke_i)K + \sum_{j=3}^{4}\tilde{g}((\tilde{\nabla}_X \tilde{J})(e_j), Ke_j)K\right\} \\ &= \frac{1}{4}\{\tilde{g}(\alpha(X, Je_1) - J^{\perp}\alpha(X, e_1), e_3)I + \tilde{g}(\alpha(X, Je_2) - J^{\perp}\alpha(X, e_2), e_4)I \\ &\quad + \tilde{g}(-A_{J^{\perp}e_3}X + JA_{e_3}X, -e_1)I + \tilde{g}(-A_{J^{\perp}e_4}X + JA_{e_4}X, -e_2)I \\ &\quad + \tilde{g}(\alpha(X, Je_1) - J^{\perp}\alpha(X, e_1), e_4)K + \tilde{g}(\alpha(X, Je_2) - J^{\perp}\alpha(X, e_2), -e_3)K \\ &\quad + \tilde{g}(-A_{J^{\perp}e_3}X + JA_{e_3}X, e_2)K + \tilde{g}(-A_{J^{\perp}e_4}X + JA_{e_4}X, e_1)K\} \\ &= \tilde{g}(\alpha(X, Je_1) - J^{\perp}\alpha(X, e_1), e_3)I + \tilde{g}(\alpha(X, Je_1) - J^{\perp}\alpha(X, e_1), e_4)K\end{aligned}$$

for all $X \in TM$. □

A surface M in $\tilde{M}$, immersed by f, is said to be *superminimal* if $\tilde{J}_*(TM) \subset f^{\#}(T^h\mathcal{Z})$, that is, $\tilde{\nabla}\tilde{J} = 0$. For superminimal surfaces, see [12,13,18], for example. The following fact is stated in [18].

Lemma 4.2. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then, M is superminimal if and only if $\alpha(X, JY) = J^{\perp}\alpha(X, Y)$ for all X and $Y \in TM$.*

Proof. Take any unit vector $e_1 \in T_xM$ at any point $x \in M$. Then, we can choose an orthonormal basis of $T_x\tilde{M}$ such that e_1, e_2 defines the orientation of M, e_3, e_4 are normal to M and $e_1, \ldots, e_4$ is compatible with the orientation of $\tilde{M}$. By Lemma 4.1, we see that $\alpha(X, Je_1) = J^{\perp}\alpha(X, e_1)$ for all $X \in T_xM$, if, and only if M is superminimal. □

If $\tilde{J}_* \circ J = J^{\mathcal{Z}} \circ \tilde{J}_*$ (precisely, $(f_{\#} \circ \tilde{J})_* \circ J = J^{\mathcal{Z}} \circ (f_{\#} \circ \tilde{J})_*$, where $f_{\#} : U(f^{\#}Q) \to UQ$ is the bundle map), then M is called a *twistor holomorphic* surface. For twistor holomorphic surfaces, see [6,12], for example.

We define a $T^{\perp}M$-valued symmetric tensor B by

$$B(X, Y) = \alpha(X, JY) - J^{\perp}\alpha(X, Y) + J^{\perp}\alpha(JX, JY) + \alpha(JX, Y) \quad (4.4)$$

for all $X, Y \in TM$. In [12], the following fact is stated in a different form.

Lemma 4.3. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then M is twistor holomorphic if and only if $B = 0$.*

Proof. From the definition of $J^{\mathcal{Z}}$, M is twistor holomorphic if and only if $J^v\tilde{\nabla}_X\tilde{J} = \tilde{\nabla}_{JX}\tilde{J}$ for all $X \in TM$. Since $J^v(I) = -K$ and $J^v(K) = I$, Lemma 4.1 gives the desired result. □

Let $\tilde{M}'$ be the manifold $\tilde{M}$ with the opposite orientation. Then both the twistor lifts of M are superminimal (resp. twistor holomorphic) for the two immersions of M into $\tilde{M}$ and $\tilde{M}'$ if and only if M is totally geodesic (resp. totally umbilic).

We define $\rho_{(e_1,e_2)}$ by

$$\rho_{(e_1,e_2)} = \tilde{g}(J^{\perp}\alpha(e_1, e_1) - \alpha(e_1, e_2), J^{\perp}\alpha(e_2, e_2) + \alpha(e_1, e_2)),$$

where e_1, e_2 is an orthonormal frame on M which is compatible with the orientation of M.

Lemma 4.4. *Let* (M, g) *be an oriented surface in an oriented four-dimensional Riemannian manifold* $(\tilde{M}, \tilde{g})$. *Then we have*

$$\|B\|^2 = 4\|\tilde{\nabla}\tilde{J}\|^2 - 8\rho_{(e_1,e_2)}. \tag{4.5}$$

Proof. We set $B_{ij} = B(e_i, e_j)$ and $\alpha_{ij} = \alpha(e_i, e_j)$ for $i, j = 1, 2$. Then we have

$$\begin{aligned}
&B_{11} = \alpha_{12} - J^{\perp}\alpha_{11} + J^{\perp}\alpha_{22} + \alpha_{12},\\
&B_{12} = -J^{\perp}(\alpha_{12} - J^{\perp}\alpha_{11}) - J^{\perp}(J^{\perp}\alpha_{22} + \alpha_{12}),\\
&B_{22} = -(\alpha_{12} - J^{\perp}\alpha_{11}) - (J^{\perp}\alpha_{22} + \alpha_{12}),\\
&\|\tilde{\nabla}\tilde{J}\|^2 = \|\alpha_{12} - J^{\perp}\alpha_{11}\|^2 + \|\alpha_{22} - J^{\perp}\alpha_{21}\|^2
\end{aligned}$$

by Lemma 4.1 and (4.4). Therefore, from the definition of $\rho_{(e_1,e_2)}$, we obtain (4.5). □

From Lemma 4.4, we see that $\rho_{(e_1,e_2)}$ does not depend on the choice of e_1, e_2. Thus, we write ρ instead of $\rho_{(e_1,e_2)}$. Note that $\rho = 0$ if M is superminimal, and the converse, in general, does not hold. If M is twistor holomorphic, then we have $\rho \geq 0$. Let $\mathcal{K}$ be the Gaussian curvature of M and $\mathcal{K}^{\perp}$ the normal curvature of $T^{\perp}M$. We have

Lemma 4.5. *Let* (M, g) *be an oriented surface in an oriented four-dimensional Riemannian manifold* $(\tilde{M}, \tilde{g})$. *Then, we have*

$$\|\tilde{\nabla}\tilde{J}\|^2 = \|\alpha\|^2 + 2\tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4) + 2\mathcal{K}^{\perp} \tag{4.6}$$

and

$$\rho = \det(A_{e_3}) + \det(A_{e_4}) - \mathcal{K}^{\perp} - \tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4). \tag{4.7}$$

Proof. We can take an orthonormal basis $e_1, \ldots, e_4$ such that $A_{e_3}(e_1) = \lambda e_1$, $A_{e_3}(e_2) = \mu e_2$, $A_{e_4}(e_1) = ae_1 + be_2$, $A_{e_4}(e_2) = be_1 + ce_2$ at each point $x \in M$. Setting $\alpha_{ij} = \alpha(e_i, e_j)$ $(i, j = 1, 2)$, we see that $\alpha_{11} = \lambda e_3 + ae_4$, $\alpha_{12} = be_4$, $\alpha_{22} = \mu e_3 + ce_4$. From Lemma 4.1, we see that

$$\begin{aligned}
\|\tilde{\nabla}\tilde{J}\|^2 &= \tilde{g}(\alpha_{12} - J^{\perp}\alpha_{11}, \alpha_{12} - J^{\perp}\alpha_{11}) + \tilde{g}(\alpha_{22} - J^{\perp}\alpha_{21}, \alpha_{22} - J^{\perp}\alpha_{21})\\
&= \tilde{g}(\alpha_{12}, \alpha_{12}) - 2\tilde{g}(\alpha_{12}, J^{\perp}\alpha_{11}) + \tilde{g}(\alpha_{11}, \alpha_{11}) + \tilde{g}(\alpha_{22}, \alpha_{22}) - 2\tilde{g}(\alpha_{22}, J^{\perp}\alpha_{21}) + \tilde{g}(\alpha_{21}, \alpha_{21})\\
&= \lambda^2 + \mu^2 + a^2 + 2b^2 + c^2 + 2b(\lambda - \mu).
\end{aligned}$$

On the other hand, we have

$$\|\alpha\|^2 = \lambda^2 + \mu^2 + a^2 + 2b^2 + c^2$$

and

$$\mathcal{K}^{\perp} = -\tilde{g}(R^{\perp}(e_1, e_2)e_3, e_4) = -\tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4) + b(\lambda - \mu)$$

by (4.3). Therefore, it holds that

$$\|\tilde{\nabla}\tilde{J}\|^2 = \|\alpha\|^2 + 2\tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4) + 2\mathcal{K}^{\perp}.$$

Similarly, we obtain

$$\rho = -(b + \lambda)(b - \mu) + ac = \det(A_{e_3}) + \det(A_{e_4}) - \mathcal{K}^{\perp} - \tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4). \quad \square$$

We set

$$\kappa := \tilde{g}(\tilde{R}(e_1, e_2)e_2, e_1) + \tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4). \tag{4.8}$$

It is easy to see that κ does not depend on the choice of the frame e_1, e_2, e_3, e_4. We note that $\tau = 12\kappa$ if $\tilde{M}$ is a self-dual Einstein manifold with the scalar curvature τ.

Lemma 4.6. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then, we have*

$$\|\tilde{\nabla}\tilde{J}\|^2 = 2\kappa + 4\|H\|^2 - 2\mathcal{K} + 2\mathcal{K}^{\perp} \tag{4.9}$$

and

$$\mathcal{K} - \mathcal{K}^{\perp} = \kappa + \rho. \tag{4.10}$$

Proof. By (4.1), we obtain

$$2\mathcal{K} = 2\tilde{g}(\tilde{R}(e_1, e_2)e_2, e_1) + 4\|H\|^2 - \|\alpha\|^2 \tag{4.11}$$

and

$$\mathcal{K} = \tilde{g}(\tilde{R}(e_1, e_2)e_2, e_1) + \det(A_{e_3}) + \det(A_{e_4}). \tag{4.12}$$

From (4.6) and (4.11), we have (4.9), and it is easy to see (4.10) by (4.7) and (4.12). □

Lemma 4.7. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then, we have*

$$\|B\|^2 = 16(\kappa + \|H\|^2 - \mathcal{K} + \mathcal{K}^{\perp}) \tag{4.13}$$

and

$$8\|\tilde{\nabla}\tilde{J}\|^2 = \|B\|^2 + 16\|H\|^2. \tag{4.14}$$

Proof. By (4.9), (4.10) and Lemma 4.4, we obtain (4.13). From (4.13) and (4.9), it is easy to obtain (4.14). □

Note that M is superminimal if and only if M is twistor holomorphic and minimal by (4.14) (see [12]). Let $\chi(M)$ (resp. $\chi(T^{\perp}M)$) be the Euler characteristic of M (resp. $T^{\perp}M$). From (4.13), we have

Corollary 4.8. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. If M is compact, we have*

$$\frac{1}{2\pi}\int_M \kappa \, dv_g + \frac{1}{2\pi}\int_M \|H\|^2 dv_g - \chi(M) + \chi(T^{\perp}M) \geq 0. \tag{4.15}$$

The equality of (4.15) holds if and only if M is twistor holomorphic.

We note that the inequality (4.15) is a generalization of Theorem 1 in [12].

Remark 1. From (4.9), (4.10), and Lemma 4.4, we have

$$\|B\|^2 = 16\|H\|^2 - 16\rho. \tag{4.16}$$

Let $\tilde{M}$ be a complex space form of constant holomorphic sectional curvature c with the orientation $\Omega \wedge \Omega$, where Ω is the fundamental form of $\tilde{M}$. If M is a Lagrangian surface in $\tilde{M}$, then we have $\mathcal{K} = -\mathcal{K}^{\perp}$. From (4.10), it follows that $\rho = 2\mathcal{K} - (1/2)c$. Hence, we have

$$\|H\|^2 \geq 2\mathcal{K} - \frac{1}{2}c. \tag{4.17}$$

The equality of (4.17) holds if and only if M is twistor holomorphic (see [6]).

5. Surfaces whose twistor lifts are harmonic sections

Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. If e_1, e_2, e_3, e_4 is an orthonormal frame adapted to the orientation of $\tilde{M}$ such that e_1, e_2 defines the orientation of M and e_3, e_4 are normal to M, then we have $\tilde{J} = f^{\#}(K_1)$. Set $I = f^{\#}(K_2)$ and $K = f^{\#}(K_3) \in \Gamma(U(f^{\#}Q))$.

Lemma 5.1. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then we have*

$$\begin{aligned}(H^{\tilde{\nabla}}(X, Y)\tilde{J})^t &= \tilde{g}(-(\nabla'_X\alpha)(Y, Je_1) + J^{\perp}(\nabla'_X\alpha)(Y, e_1), e_3)I^t \\ &\quad + \tilde{g}(-(\nabla'_X\alpha)(Y, Je_1) + J^{\perp}(\nabla'_X\alpha)(Y, e_1), e_4)K^t \qquad (5.1)\end{aligned}$$

on $\tilde{J}(M)$ for all $X, Y \in TM$, where I^t and K^t are the tangential lifts of I and K.

Proof. We may assume that $(\nabla e_1)_x = 0$, $(\nabla e_2)_x = 0$, $(\nabla^{\perp} e_3)_x = 0$ and $(\nabla^{\perp} e_4)_x = 0$ at $x \in M$. From Lemma 4.1, it follows that at $x \in M$

$$\begin{aligned}H^{\tilde{\nabla}}(X, Y)\tilde{J} &= \tilde{g}(-(\nabla'_X\alpha)(Y, Je_1) + J^{\perp}(\nabla'_X\alpha)(Y, e_1), e_3)I + \tilde{g}(-(\nabla'_X\alpha)(Y, Je_1) + J^{\perp}(\nabla'_X\alpha)(Y, e_1), e_4)K \\ &\quad - \tilde{g}(\alpha(Y, Je_1) - J^{\perp}\alpha(Y, e_1), e_3)\tilde{\nabla}_X I - \tilde{g}(\alpha(Y, Je_1) - J^{\perp}\alpha(Y, e_1), e_4)\tilde{\nabla}_X K\end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Since $\tilde{g}(\tilde{\nabla}_X I, K)_x = -\tilde{g}(\tilde{\nabla}_X K, I)_x = 0$, we have (5.1). □

We define a $T^{\perp}M$-valued 1-form $\delta\alpha$ by

$$(\delta\alpha)(X) = -\sum_{i=1}^{2}(\nabla'_{e_i}\alpha)(e_i, X)$$

for all $X \in TM$, where e_1, e_2 is an orthonormal frame of M. From (2.2) and Lemma 5.1, we have

Theorem 5.2. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. The twistor lift $\tilde{J}$ is a harmonic section if and only if it holds that $(\delta\alpha)(JX) = J^{\perp}(\delta\alpha)(X)$ for all $X \in TM$.*

Obviously, the twistor lift of a superminimal surface is a harmonic section. In addition, if M is compact, it attains its minimum value $\mathrm{Vol}(M)$ for the restricted energy functional. If M has the parallel second fundamental form, then the twistor lift of M is a harmonic section. If $(H^{\tilde{\nabla}}(X, Y)\tilde{J})^t = 0$ on $\tilde{J}(M)$ for all $X, Y \in \Gamma(TM)$, then $\tilde{J}$ is a harmonic section. By Lemma 5.1, we see that

$$(H^{\tilde{\nabla}}(X, Y)\tilde{J})^t = 0 \qquad (5.2)$$

on $\tilde{J}(M)$ for all $X, Y \in \Gamma(TM)$ if and only if

$$(\nabla'_X\alpha)(Y, JZ) = J^{\perp}(\nabla'_X\alpha)(Y, Z) \qquad (5.3)$$

for all $X, Y, Z \in \Gamma(TM)$. We define $\nabla' B$ by

$$(\nabla'_X B)(Y, Z) = \nabla^{\perp}_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. As we noted earlier, M is a superminimal surface in $\tilde{M}$ if and only if $H = 0$ and $B = 0$. Correspondingly, we have the following theorem.

Theorem 5.3. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. The twistor lift $\tilde{J}$ satisfies (5.2) on $\tilde{J}(M)$ for all $X, Y \in \Gamma(TM)$ if and only if $\nabla^{\perp} H = 0$ and $\nabla' B = 0$.*

Proof. Assume that the twistor lift $\tilde{J}$ satisfies (5.2) for all $X, Y \in \Gamma(TM)$. From (5.3), we have

$$2\nabla^{\perp}_X H = (\nabla'_X\alpha)(u, u) + (\nabla'_X\alpha)(Ju, Ju) = 0,$$

where u is a unit vector on M, and

$$\begin{aligned}(\nabla'_X B)(Y, Z) &= (\nabla'_X\alpha)(Y, JZ) - J^\perp(\nabla'_X\alpha)(Y, Z) + J^\perp(\nabla'_X\alpha)(JY, JZ) + (\nabla'_X\alpha)(JY, Z)\\ &= 0\end{aligned}$$

for all $X, Y, Z \in TM$. Conversely, we assume that $\nabla^\perp H = 0$ and $\nabla' B = 0$. Take any tangent vector Y at any point $x \in M$ and let $\tilde{Y}$ be a vector field defined on a neighborhood of $x \in M$ such that $\tilde{Y}_x = Y$ and $(\nabla\tilde{Y})_x = 0$. From $\nabla^\perp H = 0$, we have

$$(\nabla'_X\alpha)(Y, Y) + (\nabla'_X\alpha)(JY, JY) = 0$$

for all $X \in T_x M$. Therefore, by polarization, it holds that

$$(\nabla'_X\alpha)(Y, Z) + (\nabla'_X\alpha)(JY, JZ) = 0 \tag{5.4}$$

for all $X, Y, Z \in T_x M$. On the other hand, from $\nabla' B = 0$, we obtain

$$(\nabla'_X\alpha)(Y, JZ) - J^\perp(\nabla'_X\alpha)(Y, Z) + J^\perp(\nabla'_X\alpha)(JY, JZ) + (\nabla'_X\alpha)(JY, Z) = 0 \tag{5.5}$$

for all $X, Y, Z \in T_x M$. From (5.4) and (5.5), we obtain (5.3). □

Let (P, D) (resp. $(\tilde{P}, \tilde{D})$) be a smooth manifold P (resp. $\tilde{P}$) with a (not necessarily Levi-Civita) connection D (resp. $\tilde{D}$). We say that a smooth map $F : (P, D) \to (\tilde{P}, \tilde{D})$ from (P, D) to $(\tilde{P}, \tilde{D})$ is a *totally geodesic* map if

$$\tilde{D}_X F_*(Y) = F_*(D_X Y)$$

for all $X, Y \in \Gamma(TP)$. If the curvature form R^Q of Q vanishes, and if M satisfies $\nabla^\perp H = 0$ and $\nabla' B = 0$, then the twistor lift $\tilde{J}$ is a totally geodesic embedding from Lemma 2.3.

Remark 2. An immersion $F : (P, D) \to (\tilde{P}, \tilde{D})$ is said to be an *affine immersion* with transversal bundle $\mathcal{N}$ if it holds that $F^\#(T\tilde{P}) = TP \oplus \mathcal{N}$ and the induced connection of the pull back connection $F^\#\tilde{D}$ equals to D with respect to the decomposition $F^\#(T\tilde{P}) = TP \oplus \mathcal{N}$. If the curvature form R^Q of Q vanishes, then the twistor lift $\tilde{J}$ of $f : M \to \tilde{M}$ is an affine embedding with transversal bundle $\tilde{J}^\#(f^\#(T^v\mathcal{Z}))$. In [17], sections of sphere bundles are studied from the viewpoint of affine differential geometry. We refer to [19] for affine immersions.

We define δB by

$$(\delta B)(X) = -\sum_{i=1}^{2}(\nabla'_{e_i} B)(e_i, X)$$

for all $X \in TM$, where e_1, e_2 is an orthonormal frame of M. We have the following corollary.

Corollary 5.4. *Let (M, g) be an oriented surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$ such that*

$$\tilde{R}(TM, TM)(TM) \subset TM. \tag{5.6}$$

Then, the following statements are mutually equivalent:
(1) *The twistor lift $\tilde{J}$ is a harmonic section.*
(2) *The mean curvature vector H satisfies $\nabla^\perp_{JX} H = J^\perp \nabla^\perp_X H$ for all $X \in TM$.*
(3) $\delta B = 0$.

Proof. From (4.2) and (5.6), it follows that

$$(\nabla'_X\alpha)(Y, Z) = (\nabla'_Y\alpha)(X, Z)$$

for all $X, Y, Z \in TM$. Therefore, we have

$$\begin{aligned}-(\delta B)(X) &= (\nabla'_{e_1}\alpha)(e_1, JX) - J^\perp(\nabla'_{e_1}\alpha)(e_1, X) + J^\perp(\nabla'_{e_1}\alpha)(Je_1, JX) + (\nabla'_{e_1}\alpha)(Je_1, X)\\ &\quad + (\nabla'_{Je_1}\alpha)(Je_1, JX) - J^\perp(\nabla'_{Je_1}\alpha)(Je_1, X) - J^\perp(\nabla'_{Je_1}\alpha)(e_1, JX) - (\nabla'_{Je_1}\alpha)(e_1, X)\\ &= -(\delta\alpha)(JX) + J^\perp(\delta\alpha)(X)\end{aligned}$$

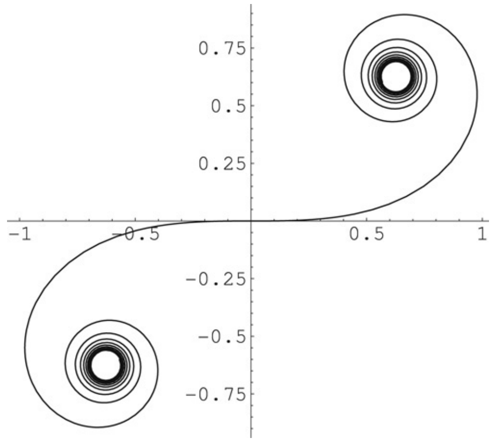

Fig. 1. Polynomial spiral with the curvature function $2u$.

for all $X \in TM$. Then, (1) is equivalent to (3). Since it holds that $\nabla^{\perp}_X H = -(\delta\alpha)(X)$ for $X \in TM$, we see that (1) is equivalent to (2). □

The normal connection $\nabla^{\perp}$ on the normal bundle defines a holomorphic structure such that $\zeta \in \Gamma(T^{\perp}M)$ is holomorphic if and only if it holds that $\nabla^{\perp}_{JX}\zeta = J^{\perp}\nabla^{\perp}_X\zeta$ for all $X \in TM$. Therefore, the statement (2) in Corollary 5.4 is equivalent to the holomorphicity of the mean curvature vector $H \in \Gamma(T^{\perp}M)$. In the case where $\tilde{M}$ is a space form of constant curvature, or M is a Lagrangian surface in a complex space form $\tilde{M}$ of constant holomorphic sectional curvature, the condition (5.6) is satisfied. If M is an invariant surface in the complex space form $\tilde{M}$, then the twistor lift is a harmonic section. In fact, such surfaces satisfy (5.6) and $H = 0$. In [7,11], the usual harmonicities of twistor lifts are considered.

Example 1. Let $\gamma_i : I_i \to \mathbf{R}^2$ be a smooth curve with arc length parameter in $\mathbf{R}^2$, where I_i is an open interval $(i = 1, 2)$. We denote the tangent vector of γ_i by T_i, and the normal vector by N_i such that $\det(T_i \ N_i) = 1$ $(i = 1, 2)$. We consider the product surface M in $\mathbf{R}^2 \times \mathbf{R}^2 \simeq \mathbf{R}^4$ given by $(s, t) \mapsto (\gamma_1(s), \gamma_2(t))$. Take the orthonormal frame

$$e_1 = (T_1, 0), \qquad e_2 = (0, T_2), \qquad e_3 = \frac{1}{\sqrt{2}}(N_1, N_2), \qquad e_4 = \frac{1}{\sqrt{2}}(N_1, -N_2).$$

By a straightforward calculation, we have

$$H = -\frac{1}{2\sqrt{2}}(\kappa_1 + \kappa_2)e_3 - \frac{1}{2\sqrt{2}}(\kappa_1 - \kappa_2)e_4,$$

where κ_i is the curvature of γ_i. Then the twistor lift of M is a harmonic section if and only if

$$\frac{\mathrm{d}}{\mathrm{d}s}\kappa_1 = \frac{\mathrm{d}}{\mathrm{d}t}\kappa_2. \tag{5.7}$$

Since two sides in (5.7) depend on different variables, we have $\kappa_i(u) = cu + d_i$ $(i = 1, 2)$, where c, d_1, d_2 are constants. A plane curve is called a *polynomial spiral* if its curvature function is a polynomial function of the arc length parameter. Therefore the twistor lift of the product surface M in $\mathbf{R}^4$ is a harmonic section if and only if M is the product surface of polynomial spirals γ_1 and γ_2 such that $\kappa_i(u) = cu + d_i$ $(i = 1, 2)$. In particular, if $c = 0$, then the polynomial spiral is an open part of a circle or a line. Fig. 1 shows the polynomial spiral with $c \neq 0$. If $c \neq 0$, then $\|\tilde{\nabla}\tilde{J}\|^2$ is not constant. In Section 6, we consider compact surfaces such that the twistor lifts are harmonic sections and the energy densities of the twistor lifts are constant.

Remark 3. Let $\tilde{M}'$ be the manifold $\tilde{M}$ with the opposite orientation. Then, we see that both the twistor lifts for two immersions into $\tilde{M}$ and $\tilde{M}'$ satisfy (5.2) if and only if M has the parallel second fundamental form. Similarly, both twistor lifts are harmonic sections if and only if $\delta\alpha = 0$.

Let $\tilde{M}$ be a four-dimensional hyperkähler manifold and I_1, I_2, I_3 a hyperkähler structure on $\tilde{M}$. If the orientation of $\tilde{M}$ is given by

$$-\sum_{i=1}^{3} \Omega_{I_i} \wedge \Omega_{I_i},$$

then we have $I_1, I_2, I_3 \in \Gamma(\mathcal{Z})$, where Ω_{I_i} is the two form defined by $\Omega_{I_i}(X, Y) = \tilde{g}(I_i X, Y)$ for $X, Y \in TM$ ($i = 1, 2, 3$). The twistor space $\mathcal{Z}$ of $\tilde{M}$ is $\tilde{M} \times S^2(1)$, where $S^2(1)$ is the two dimensional unit sphere. Let $\hat{p} : \mathcal{Z} \to S^2(1)$ be the projection. For a surface M in $\tilde{M}$, we set $\tilde{J}' := \hat{p} \circ \tilde{J}$ ($=\hat{p} \circ f_\# \circ \tilde{J}$, precisely). We have the following theorem:

Theorem 5.5. *Let (M, g) be an oriented surface in a four-dimensional hyperkähler manifold $(\tilde{M}, \tilde{g})$. Then the following statements are mutually equivalent:*

(1) *The twistor lift $\tilde{J}$ is a harmonic section.*
(2) *The twistor lift $\tilde{J}$ is a harmonic map in the usual sense.*
(3) *$\tilde{J}'$ is a harmonic map in the usual sense.*

Proof. It is trivial that (1) is equivalent to (2) in view of Lemma 2.4. The map $\tilde{J}'$ is explicitly given by

$$\tilde{J}' = (\tilde{g}(I_1, \tilde{J}), \tilde{g}(I_2, \tilde{J}), \tilde{g}(I_3, \tilde{J})).$$

Let D be the Levi-Civita connection on $\mathbf{R}^3$ with respect to the standard metric and $\bar{\nabla}$ the induced connection on $S^2(1)$. Since

$$\tilde{J}'_*(X) = (\tilde{g}(I_1, \tilde{\nabla}_X \tilde{J}), \tilde{g}(I_2, \tilde{\nabla}_X \tilde{J}), \tilde{g}(I_3, \tilde{\nabla}_X \tilde{J}))$$

for all $X \in TM$, we have

$$D_X \tilde{J}'_* Y = (\tilde{g}(I_1, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J}), \tilde{g}(I_2, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J}), \tilde{g}(I_3, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J}))$$

for all $X, Y \in \Gamma(TM)$. Therefore it holds that

$$\bar{\nabla}_X \tilde{J}'_* Y = (\tilde{g}(I_1, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J}), \tilde{g}(I_2, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J}), \tilde{g}(I_3, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J})) - \tilde{g}(\tilde{J}, \tilde{\nabla}_X \tilde{\nabla}_Y \tilde{J})\tilde{J}'$$

for all $X, Y \in \Gamma(TM)$. Hence, the torsion $\tau(\tilde{J}')$ of $\tilde{J}'$ is given by

$$\begin{aligned}\tau(\tilde{J}') &= (\tilde{g}(I_1, \bar{\Delta}^{\tilde{\nabla}} \tilde{J}), \tilde{g}(I_2, \bar{\Delta}^{\tilde{\nabla}} \tilde{J}), \tilde{g}(I_3, \bar{\Delta}^{\tilde{\nabla}} \tilde{J})) - \tilde{g}(\tilde{J}, \bar{\Delta}^{\tilde{\nabla}} \tilde{J})\tilde{J}' \\ &= (\tilde{g}(I_1, \bar{\Delta}^{\tilde{\nabla}} \tilde{J} - \|\tilde{\nabla} \tilde{J}\| \tilde{J}), \tilde{g}(I_2, \bar{\Delta}^{\tilde{\nabla}} \tilde{J} - \|\tilde{\nabla} \tilde{J}\| \tilde{J}), \tilde{g}(I_3, \bar{\Delta}^{\tilde{\nabla}} \tilde{J} - \|\tilde{\nabla} \tilde{J}\| \tilde{J})).\end{aligned}$$

Then, we see that the twistor lift $\tilde{J}$ is a harmonic section if and only if $\tilde{J}'$ is a harmonic map in the usual sense. □

For a surface M in a four-dimensional hyperkähler manifold $\tilde{M}$, the degree of the map $\tilde{J}' : M \to S^2(1)$ is denoted by $\deg(\tilde{J}')$. The degree $\deg(\tilde{J}')$ is related to ρ as follows:

Lemma 5.6. *Let (M, g) be an oriented compact surface in a four-dimensional hyperkähler manifold $(\tilde{M}, \tilde{g})$. Then we have*

$$\int_M \rho \, \mathrm{d}v_g = 4\pi \deg(\tilde{J}').$$

Proof. Let $\bar{\omega}$ be the standard volume element on $S^2(1)$. Set $\bar{\omega}' = 1/(4\pi)\bar{\omega}$. Since $(\tilde{J}'^* \bar{\omega})(e_1, e_2) = -\tilde{g}(\alpha(e_1, e_2) - J^\perp \alpha(e_1, e_1), J^\perp \alpha(e_2, e_2) + \alpha(e_1, e_2)) = \rho$ for an orthonormal frame e_1, e_2 on M, which is compatible with the orientation, it follows that

$$\deg(\tilde{J}') = \int_M \tilde{J}'^* \bar{\omega}' = \frac{1}{4\pi} \int_M (\tilde{J}'^* \bar{\omega})(e_1, e_2) \mathrm{d}v_g = \frac{1}{4\pi} \int_M \rho \, \mathrm{d}v_g$$

from Lemma 4.1. □

Let M be a compact twistor holomorphic surface in a four-dimensional hyperkähler manifold $\tilde{M}$. Since the projection $\hat{p} : \mathcal{Z} \to S^2(1)$ is holomorphic, the map $\tilde{J}' : M \to S^2(1)$ is holomorphic. If M is not superminimal, then the superminimal points of M coincide with the branch points of $\tilde{J}'$. Let $p_1, \dots, p_l$ be the branch points of $\tilde{J}'$ and $r_1, \dots, r_l$ their degrees of ramification. From the Riemann–Hurwitz relation, we have

$$\chi(M) + \sum_{i=1}^{l} r_i = 2\deg(\tilde{J}'). \tag{5.8}$$

On the other hand, it holds that

$$2\deg(\tilde{J}') = \chi(M) - \chi(T^{\perp}M) \tag{5.9}$$

by Lemma 5.6 and (4.10). Then, we have

$$\chi(T^{\perp}M) = -\sum_{i=1}^{l} r_i \ (\leq 0)$$

from (5.8) and (5.9). We refer to [12] for the case when $\tilde{M} = \mathbf{R}^4$. In connection with the fact that $\chi(T^{\perp}M) \leq 0$ for a twistor holomorphic surface, we have

Corollary 5.7. *Let (M, g) be an oriented compact surface with genus q in a hyperkähler manifold $(\tilde{M}, \tilde{g})$. If the twistor lift $\tilde{J}$ is a harmonic section, and*

$$2(1 - 2q) \geq \chi(T^{\perp}M),$$

then M is a twistor holomorphic surface in $\tilde{M}$.

Proof. From Lemma 5.6 and (4.10), we have $4\pi \deg(\tilde{J}') = 4\pi(1 - q) - 2\pi\chi(T^{\perp}M)$. Then, it holds that

$$\deg(\tilde{J}') = (1 - q) - \frac{1}{2}\chi(T^{\perp}M) \geq (1 - q) - \frac{1}{2} \cdot 2(1 - 2q) = q.$$

By Theorem 5.5, $\tilde{J}'$ is a harmonic map with $\deg(\tilde{J}') \geq q$. Every harmonic map φ from an oriented surface to $S^2(1)$ with $\deg(\varphi) \geq q$ is holomorphic [10]. Then, we have $\tilde{J}'_* J = \bar{J}\tilde{J}'_*$, where $\bar{J}$ is the complex structure on $S^2(1)$. Because of $\hat{p}_* J^{\mathcal{Z}} = \bar{J}\hat{p}_*$, it holds that

$$\hat{p}_* J^{\mathcal{Z}} \tilde{J}_* = \bar{J}\hat{p}_*\tilde{J}_* = \bar{J}\tilde{J}'_* = \tilde{J}'_* J = \hat{p}_*\tilde{J}_* J.$$

Therefore, $J^{\mathcal{Z}}\tilde{J}_*(X) - (\tilde{J}_* J(X))$ is horizontal, that is, $J^{\mathcal{Z}}(\tilde{\nabla}_X \tilde{J}) = \tilde{\nabla}_{JX}\tilde{J}$ for all $X \in TM$. □

Remark 4. From Lemmas 4.4 and 5.6, we can see a quantization phenomenon

$$\frac{1}{8\pi}\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g - \frac{1}{32\pi}\int_M \|B\|^2 \mathrm{d}v_g = \deg(\tilde{J}') \in \mathbf{Z}$$

for surfaces in four-dimensional hyperkähler manifolds. In particular, the twistor lift for any twistor holomorphic surface satisfies

$$\frac{1}{8\pi}\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g \in \mathbf{N} \cup \{0\}.$$

If $\tilde{M} = \mathbf{R}^4$ and M is a compact twistor holomorphic surface such that

$$\frac{1}{8\pi}\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g = 1, \tag{5.10}$$

then M is the standard sphere. In fact, if Eq. (5.10) holds, then we have $\deg(\tilde{J}') = 1$. From (4.9), (4.10), and Lemma 5.6, we obtain

$$\int_M \|H\|^2 \mathrm{d}v_g = 4\pi.$$

Therefore, M is the standard sphere (see [8]).

6. The energy density of the twistor lifts for surfaces in Euclidean space

The vertical component

$$\frac{1}{2}\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g \quad \left(\text{resp. } \frac{1}{2}\|\tilde{\nabla}\tilde{J}\|^2\right)$$

of energy $\mathcal{E}(\tilde{J})$ (resp. energy density) for the twistor lift is called the vertical energy (resp. the vertical energy density). We see that M is a superminimal surface in $\tilde{M}$ if and only if $\|\tilde{\nabla}\tilde{J}\|^2 = 0$. Hence, it is natural to consider the following problem : *Assume that $\tilde{M}$ does not admit any compact superminimal surface. Find a geometric constant $C > 0$ such that*

$$\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g \geq C$$

and characterize the equality case. In this section, we consider this problem in the case when $\tilde{M}$ is the four-dimensional Euclidean space $\mathbf{R}^4$, and study surfaces in $\mathbf{R}^4$ such that the twistor lifts are harmonic sections, and the vertical energy density of the twistor lifts are constant. Let Δ be the Laplacian acting on the smooth functions on M and $\lambda_i(M)$ the i-th eigenvalues of Δ. The set of all nonzero eigenvalues of Δ is denoted by $\sigma(\Delta)$. We then have the following theorem.

Theorem 6.1. *Let M be an oriented connected compact surface in $\mathbf{R}^4$. Then we have*

$$\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g \geq \lambda_1(M)\mathrm{Vol}(M).$$

The equality holds if and only if M is the standard sphere in $\mathbf{R}^4$.

To prove Theorem 6.1, we need the following lemma:

Lemma 6.2. *Let (M, g) be an oriented compact surface in an oriented four-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$ of constant curvature c. If $\chi(T^{\perp}M) = 0$, then the following statements are mutually equivalent:*

(1) *M is twistor holomorphic.*

(2) *M is totally umbilic.*

Proof. Assume that M is twistor holomorphic. Since $\chi(T^{\perp}M) = 0$, we have

$$\frac{c}{2\pi}\mathrm{Vol}(M) + \frac{1}{2\pi}\int_M \|H\|^2 \mathrm{d}v_g - \chi(M) = 0$$

by Corollary 4.8. Because $\tilde{M}$ has the constant curvature c, $\tilde{M}$ is self-dual with respect to both orientations. Hence, we see that M is twistor holomorphic with respect to the opposite orientation of $\tilde{M}$ by Corollary 4.8, that is, M is twistor holomorphic relative to both orientations of $\tilde{M}$. Then, M is totally umbilic. The converse is trivial by Lemma 4.3. □

Here, we give the proof of Theorem 6.1.

Proof of Theorem 6.1. From (4.14), we have

$$\begin{aligned}\int_M \|\tilde{\nabla}\tilde{J}\|^2 \mathrm{d}v_g &= \frac{1}{8}\int_M \|B\|^2 v_g + 2\int_M \|H\|^2 v_g \\ &\geq 2\int_M \|H\|^2 v_g \\ &\geq \lambda_1(M)\mathrm{Vol}(M).\end{aligned}$$

To obtain the latter inequality, we use a result of [21]. By Lemma 4.3 and [21], the equality holds if and only if M is twistor holomorphic and is a minimal hypersurface in a hypersphere of certain radius in $\mathbf{R}^4$. Since a hypersphere is totally umbilic in $\mathbf{R}^4$, we have $\chi(T^{\perp}M) = 0$. Therefore, by Lemma 6.2, M is totally umbilic, that is, M is the standard sphere in $\mathbf{R}^4$. □

Theorem 6.1 leads to the study of a relation between the energy (density) and $\sigma(\Delta)$ for a surface in $\mathbf{R}^4$. For the energy density of the twistor lift $\tilde{J}$, if $\tilde{J}$ is a harmonic section and $\|\tilde{\nabla}\tilde{J}\|^2$ is constant, then $\|\tilde{\nabla}\tilde{J}\|^2$ is an eigenvalue of the rough Laplacian $\bar{\Delta}^{\tilde{\nabla}}$. In particular, if the ambient space is hyperkählerian, then $\|\tilde{\nabla}\tilde{J}\|^2$ is an intrinsic quantity of M. We have the following lemma:

Lemma 6.3. *Let (M, g) be an oriented compact surface in a four dimensional hyperkähler manifold $(\tilde{M}, \tilde{g})$. If $\tilde{J}$ is a harmonic section and $\|\tilde{\nabla}\tilde{J}\|^2$ is constant, then $\|\tilde{\nabla}\tilde{J}\|^2 \in \sigma(\Delta) \cup \{0\}$. In particular, if $\tilde{J}$ is a harmonic section, $\|\tilde{\nabla}\tilde{J}\|^2$ is constant and $\|\tilde{\nabla}\tilde{J}\|^2 < \lambda_1(M)$, then M is superminimal.*

Proof. Since $\tilde{M}$ is a hyperkähler manifold, there exists a parallel complex structure $I \in \Gamma(\mathcal{Z})$ such that $\tilde{g}(I, \tilde{J}) \neq 0$. We set $a := \tilde{g}(I, \tilde{J})$. It holds that $\Delta a = \|\tilde{\nabla}\tilde{J}\|^2 a$, since $\tilde{J}$ is a harmonic section. Then, a is an eigenfunction of Δ and $\|\tilde{\nabla}\tilde{J}\|^2 \in \sigma(\Delta) \cup \{0\}$. If $\|\tilde{\nabla}\tilde{J}\|^2 < \lambda_1(M)$, then $\tilde{\nabla}\tilde{J} = 0$, that is, M is superminimal. □

Here, we give examples such that $\tilde{J}$ is a harmonic section and $\|\tilde{\nabla}\tilde{J}\|^2$ is constant. Let $S^k(c)$ be the k-dimensional sphere with radius c.

Example 2. $\lambda_1(M) = \|\tilde{\nabla}\tilde{J}\|^2$: The totally umbilic surface $f : S^2(c) \to \mathbf{R}^4$ satisfies the conditions that the twistor lift $\tilde{J}$ is a harmonic section, $\|\tilde{\nabla}\tilde{J}\|^2$ is constant, and $\lambda_1(S^2(c)) = \|\tilde{\nabla}\tilde{J}\|^2$.

Next, we consider the canonical product surface $f_{a,b} : S^1(a) \times S^1(b) \to \mathbf{R}^4 (a, b > 0)$. We define $F_{a,b} : \mathbf{R}^2 \to \mathbf{R}^4$ by

$$F_{a,b}(x, y) = \left(a \cos \frac{x}{a}, a \sin \frac{x}{a}, b \cos \frac{y}{b}, b \sin \frac{y}{b}\right).$$

Let $\Lambda_{a,b}$ be the lattice of $\mathbf{R}^2$, which is the $\mathbf{Z}$-span by $(2\pi a, 0)$ and $(0, 2\pi b)$, and set $T^2_{a,b} := \mathbf{R}^2/\Lambda_{a,b} \cong S^1(a) \times S^1(b)$. Then, the immersion $F_{a,b}$ induces $f_{a,b}$. The twistor lift $\tilde{J}$ is a harmonic section (see Example 1). Moreover, we obtain

$$\|\tilde{\nabla}\tilde{J}\|^2 = \frac{1}{a^2} + \frac{1}{b^2}.$$

On the other hand, the dual lattice $\Lambda^*_{a,b}$ of $\Lambda_{a,b}$ is the $\mathbf{Z}$-span by $(a', 0)$ and $(0, b')$, where $a' = 1/(2\pi a)$ and $b' = 1/(2\pi b)$. The spectrum set of the Laplacian of $T_{a,b}$ is $\{4\pi^2\|x\|^2 \mid x \in \Lambda^*_{a,b}\}$. We note that $f_{a,b}$ is not twistor holomorphic, and $\rho = 0$.

Example 3. $\lambda_2(M) = \|\tilde{\nabla}\tilde{J}\|^2$: It is easy to see that $\lambda_2(T^2_{a,a}) = 2/a^2$ (see Fig. 2). Then, $f_{a,a}$ satisfies the identity $\lambda_2(T^2_{a,a}) = \|\tilde{\nabla}\tilde{J}\|^2$.

Example 4. $\lambda_i(M) = \|\tilde{\nabla}\tilde{J}\|^2$ $(i \geq 3)$: If $a \neq b$, we may assume $a > b$. Set $j = [b'/a']$, where $[x]$ stands for the maximum integer which does not exceed $x \in \mathbf{R}$. Then, we have $\lambda_i(T^2_{a,b}) = 1/a^2 + 1/b^2$, where $i = j + 1$ if $b' = ja'$, and $i = j + 2$ if $b' \neq ja'$ (see Fig. 2). Then, $f_{a,b}$ satisfies the identity $\lambda_i(T^2_{a,b}) = \|\tilde{\nabla}\tilde{J}\|^2$.

These examples are spherical surfaces; that is, they are contained in a hypersphere in $\mathbf{R}^4$. All spherical surfaces have vanishing normal curvatures. We obtain the following theorem:

Theorem 6.4. *Let M be an oriented, compact, and connected surface in $\mathbf{R}^4$ with*

$$\int_M \|H\|^2 \mathcal{K}^{\perp} \mathrm{d}v_g = 0.$$

Assume that $\tilde{J}$ is a harmonic section, and $\|\tilde{\nabla}\tilde{J}\|^2$ is constant. Then, we have the following:

(1) *If $\|\tilde{\nabla}\tilde{J}\|^2 = \lambda_1(M)$, then M is the standard sphere $S^2(a)$ in $\mathbf{R}^4$.*
(2) *If $\|\tilde{\nabla}\tilde{J}\|^2 = \lambda_2(M)$, then M is the product surface $S^1(a) \times S^1(a)$ in $\mathbf{R}^4$.*
(3) *If $\|\tilde{\nabla}\tilde{J}\|^2 = \lambda_i(M)$ $(i \geq 3)$, then M is the product surface $S^1(a) \times S^1(b)$ in $\mathbf{R}^4$, with $a \neq b$.*

For proving Theorem 6.4, we need the following lemmas.

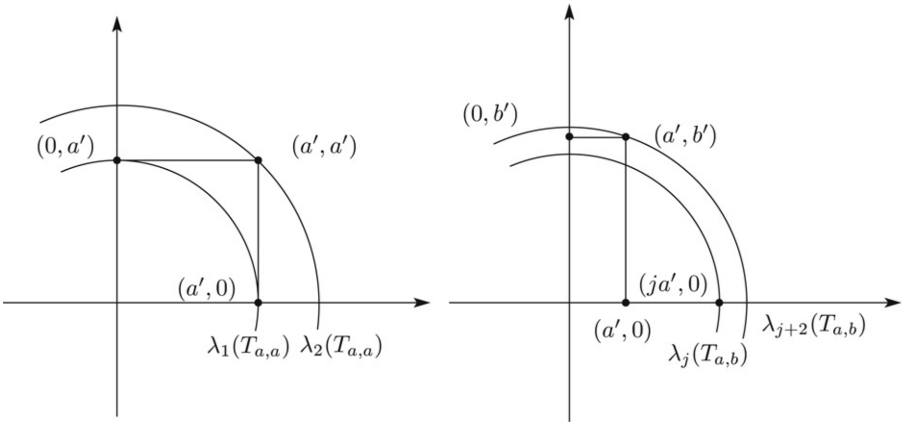

Fig. 2. Eigenvalues of the Laplacian on $T_{a,b}$.

Lemma 6.5. *Let (M, g) be an oriented surface in $(\tilde{M}, \tilde{g})$ such that*

$$\tilde{R}(TM, TM)TM \subset TM.$$

If the twistor lift $\tilde{J}$ of M is a harmonic section, then we have

$$\bar{\Delta}^{\nabla^\perp} H = -\mathcal{K}^\perp H, \tag{6.1}$$

where $\bar{\Delta}^{\nabla^\perp}$ is the rough Laplacian of the normal connection $\nabla^\perp$.

Proof. By Corollary 5.4, the mean curvature vector H is a holomorphic section of $T^\perp M$. For any local unit vector field u of M, we have

$$\begin{aligned}\bar{\Delta}^{\nabla^\perp} H &= -\nabla^\perp_u \nabla^\perp_u H + \nabla^\perp_{\nabla_u u} H - \nabla^\perp_{Ju} \nabla^\perp_{Ju} H + \nabla^\perp_{\nabla_{Ju} Ju} H \\ &= H^{\nabla^\perp}(u, u)H + J^\perp H^{\nabla^\perp}(Ju, u)H. \end{aligned} \tag{6.2}$$

We replace u by Ju in (6.2). Then, it follows that

$$\bar{\Delta}^{\nabla^\perp} H = H^{\nabla^\perp}(Ju, Ju)H - J^\perp H^{\nabla^\perp}(u, Ju)H. \tag{6.3}$$

Using (6.2) and (6.3), we have

$$2\bar{\Delta}^{\nabla^\perp} H = \bar{\Delta}^{\nabla^\perp} H + J^\perp H^{\nabla^\perp}(Ju, u)H - J^\perp H^{\nabla^\perp}(u, Ju)H.$$

Then, we obtain $\bar{\Delta}^{\nabla^\perp} H = R^\perp(u, Ju)J^\perp H$. □

Lemma 6.6. *Let (M, g) be an oriented compact surface in $(\tilde{M}, \tilde{g})$. We assume*

$$\int_M \|H\|^2 \mathcal{K}^\perp \mathrm{d}v_g = 0$$

and $\tilde{R}(TM, TM)TM \subset TM$. Then, the twistor lift $\tilde{J}$ is a harmonic section if and only if $\nabla^\perp H = 0$.

Proof. Assume that $\tilde{J}$ is a harmonic section. From Lemma 6.5, it follows that

$$\int_M \tilde{g}(\nabla^\perp H, \nabla^\perp H)\mathrm{d}v_g = 0.$$

Then the mean curvature vector is parallel with respect to $\nabla^\perp$. The converse is trivial. □

Lemma 6.7. *Let (M, g) be an oriented surface in the real space form $(\tilde{M}, \tilde{g})$, and let $\bar{M}$ be a totally umbilic hypersurface in $\tilde{M}$. Assume that M is contained in $\bar{M}$, $\|\tilde{\nabla}\tilde{J}\|^2$ is constant and the twistor lift $\tilde{J}$ of M in $\tilde{M}$ is a harmonic section. Then, M is an isoparametric surface in $\bar{M}$.*

Proof. Let ξ be the unit normal vector field on M in $\bar{M}$ and η the unit normal vector field on $\bar{M}$ in $\tilde{M}$. We set $e_3 = \xi$ and $e_4 = \eta|_M$. The mean curvature vector on M in $\bar{M}$ is denoted by $\bar{H}$. Then, we have $H = \bar{H} + \nu^2 e_4$, where ν is the mean curvature function on $\bar{M}$ in $\tilde{M}$. Since $\bar{M}$ is totally umbilic in $\tilde{M}$, we see that $\nabla^{\perp}_X e_3 = 0$, $\nabla^{\perp}_X e_4 = 0$ and ν is constant. Hence, we have

$$\nabla^{\perp}_X H = \nabla^{\perp}_X \bar{H} = \bar{\nabla}^{\perp}_X \bar{H}$$

for all $X \in TM$, where $\bar{\nabla}^{\perp}$ is the normal connection of M in $\bar{M}$. Therefore, $J^{\perp}\bar{\nabla}^{\perp}_X \bar{H}$ is proportional to e_4. By Corollary 5.4, $\tilde{J}$ is a harmonic section if, and only if $\bar{\nabla}^{\perp}_X \bar{H} = 0$ for all $X \in TM$. Let λ, μ be the principal curvatures of M in $\bar{M}$. We see that $\lambda + \mu$ is constant by $\bar{\nabla}^{\perp}\bar{H} = 0$. From (4.6), we have

$$\|\tilde{\nabla}\tilde{J}\|^2 = \lambda^2 + \mu^2 + 2\nu^2.$$

Since $\|\tilde{\nabla}\tilde{J}\|^2$ is constant, $\lambda^2 + \mu^2$ is also constant. Therefore, M is an isoparametric surface in $\bar{M}$. □

Using Lemmas 6.6 and 6.7, we can give the proof of Theorem 6.4.

Proof of Theorem 6.4. Since $\tilde{J}$ is a harmonic section and M is compact, we have $\nabla^{\perp}H = 0$ by Lemma 6.6. Therefore, by [24], M is one of the following surfaces: (1) M is a minimal surface in $\mathbf{R}^4$, (2) M is a constant mean curvature hypersurface in $\mathbf{R}^3$ or $S^3(c)$. Since M is compact, the first case does not occur. By Lemma 6.7, M is an isoparametric hypersurface in $\mathbf{R}^3$ or $S^3(c)$. Since M is compact, we obtain the desired conclusion. □

Consider the totally umbilic surface M with radius r in $S^4(1)$. Then, we see that $\|\tilde{\nabla}\tilde{J}\|^2 \notin \sigma(\Delta)$. In fact, we have $\|\tilde{\nabla}\tilde{J}\| = 2(1-r^2)/r^2$ and $\lambda_i(M) = i(i+1)/r^2$. It is easy to see that there is no positive integer i such that $2(1-r^2)/r^2 = i(i+1)/r^2$. Therefore, in Theorem 6.4, the ambient space $\mathbf{R}^4$ cannot be replaced by $S^4(1)$.

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